The theory of spectral Mackey functors

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Equivariant homotopy theory: spaces

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For a more refined invariant, can instead invert maps that induce weak equivalences on all fixed points $\rightsquigarrow$ the ∞-category of G-spaces $\mathbf{Top}_G := \mathrm{Fun}(\mathbf{O}_G^{op}, \mathbf{Top})$ (Elmendorf's theorem). Here $\mathbf{O}_G^{op}$ is the orbit category of G.

Flavors of G-spectra

Naive option for G-spectra: stabilize $\mathbf{Top}_G$ to get $\mathbf{Sp}_G^{\text{naive}} := \text{Fun}(\mathbf{O}_G^{op}, \mathbf{Sp})$. Here $\mathbf{Sp}$ is the ∞-category of spectra.

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Example: Let KU_G be equivariant complex K-theory. Evaluating KU_G on a G-orbit G/H gives $\mathrm{Rep}(H)$. $\mathrm{Rep}(-)$ enjoys additional covariant functoriality in $\mathbf{O}_G$ given by induction of representations.

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Borel option for G-spectra: take $\mathrm{Fun}(BG, \mathbf{Sp})$. Doesn't contain example of KU_G: difference measured by Atiyah-Segal completion theorem.

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- objects: same as $\mathbf{F}_G$.
- morphisms: $\mathrm{Map}(U,V) := \iota((\mathbf{F}_G)_{/(U \times V)})$, where ι denotes the maximal subgroupoid functor (discard the non-equivalences). Composition is given by forming the pullback of spans.

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Observe that $\mathrm{Span}(\mathbf{F}_G)$ admits direct sums, given by taking the coproduct of underlying G-sets. Define the ∞-category of *G-spectral Mackey functors*

$$\mathbf{Sp}_G := \mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathbf{F}_G), \mathbf{Sp})$$

to be the category of direct-sum preserving functors from $\mathrm{Span}(\mathbf{F}_G)$ to $\mathbf{Sp}$.

Another name for $\mathrm{Span}(\mathbf{F}_G)$ is the *effective Burnside category*. The Burnside (∞-)category $A(G)$ is then obtained by group completing the hom-groupoids with respect to disjoint union.

G-spectra as spectral Mackey functors

Alternative definition of G-spectra: invert representation spheres (one point compactifications of real G-representations). By a cofinality argument, it suffices to invert regular representation sphere $S^\rho \rightsquigarrow$

$$\mathbf{Sp}_G^{\text{genuine}} := \varprojlim \left(\ldots \xrightarrow{\Omega^\rho} \mathbf{Top}_{G,*} \xrightarrow{\Omega^\rho} \mathbf{Top}_{G,*} \right)$$

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Theorem (Guillou-May): There is an equivalence $\mathbf{Sp}_G^{\text{genuine}} \simeq \mathbf{Sp}_G$, where

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This other description emphasizes that one has the subgroup $\mathrm{RO}(G)$ of the Picard group $\mathrm{Pic}(\mathbf{Sp}_G)$ of invertible objects in G-spectra. On the other hand, the spectral Mackey functor approach highlights the transfers, and also separates the complexity of $\mathbf{Sp}_G$ into two parts: $\mathrm{Span}(\mathbf{F}_G)$ and $\mathbf{Sp}$.

Examples of spectral Mackey functors

Algebraic: ordinary Mackey functor $\mathrm{Span}(\mathbf{F}_G) \longrightarrow \mathbf{Ab}$ supplies G-spectrum, postcomposing by Eilenberg-Maclane functor $H : \mathbf{Ab} \longrightarrow \mathbf{Sp}$. Of course, the same thing works for chain complexes of R-modules.

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Categorical / K-theoretic: Loosely, $\mathrm{Span}(\mathbf{F}_G) \longrightarrow \mathbf{Cat}_\infty \rightsquigarrow G$-spectrum by application of some K-theory machine. E.g. produce KU_G from $G/H \mapsto \mathrm{Vect}_H$.

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In general, difficult to write down ∞-functors by hand, because one needs to specify an infinite hierarchy of coherence data. When the target is $\mathbf{Cat}_\infty$, one has an adaptation of the technology of (co)cartesian fibrations which allows one to produce categorical input.

Example of representable functors

Have Yoneda map

$$j : \mathrm{Span}(\mathbf{F}_G) \longrightarrow P_\Sigma(\mathrm{Span}(\mathbf{F}_G)) \simeq \mathrm{Fun}^\times(\mathrm{Span}(\mathbf{F}_G), \mathbf{Top})$$

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Further compose with left adjoint to levelwise Ω^∞ to get into $\mathbf{Sp}_G$. Then formula for 1-excisive approximation of a functor shows that

$$j(U) : V \mapsto K(\mathrm{Map}(U, V)) = K(\iota(\mathbf{F}_G)_{/U \times V})$$

where K is group completion. Identifying $j(U)$ with $\Sigma^\infty_+ U$, this is the equivariant Barrett-Priddy-Quillen theorem (key detail in Guillou-May comparison).

Functoriality

Restriction, induction, coinduction: The adjunction

$$\mathrm{Ind}_H^G\colon \mathbf{F}_H \simeq (\mathbf{F}_G)_{/(G/H)} \rightleftarrows \mathbf{F}_G : \mathrm{Res}_H^G$$

obtains, using self-duality of $\mathrm{Span}(-)$, the ambidextrous adjunction

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Fixed point functors: Let $H \trianglelefteq G$. Adjunction $\pi^*\colon \mathbf{F}_{G/H} \rightleftarrows \mathbf{F}_G :\pi_* \rightsquigarrow$ categorical fixed points $\Psi^H = \pi^{**} : \mathbf{Sp}_G \longrightarrow \mathbf{Sp}_{G/H}$ and geometric fixed points $\Phi^H = (\pi_*)_! : \mathbf{Sp}_G \longrightarrow \mathbf{Sp}_{G/H}$.

Upshot: target being **Sp** is immaterial for defining these functors.

Day convolution

Let C be a symmetric monoidal ∞-category and D be a presentably symmetric monoidal ∞-category ($:= D$ presentable, in particular admits all colimits, and $\otimes$ preserves colimits separately in each variable). We want a symmetric monoidal structure on $\mathrm{Fun}(C, D)$ such that:

1. $\mathrm{Fun}(C, D)$ is presentably symmetric monoidal.
2. $C^{op} \times D \xrightarrow{j \times id} \mathrm{Fun}(C, \mathbf{Top}) \times D \xrightarrow{\odot} \mathrm{Fun}(C, D)$ is symmetric monoidal (generalizes, when $D = \mathbf{Top}$, requiring that Yoneda map is symmetric monoidal).

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Explicitly, given two functors $F, G : C \longrightarrow D$, $F \otimes G$ computed as the left Kan extension $(F \otimes G)(c) = \underset{(c_1 \times c_2 \to c) \in C_{/c} \times_C C \times C}{\mathrm{colim}} F(c_1) \otimes_D G(c_2)$,

$$\begin{tikzcd} C \times C \arrow[r, "F \times G"] \arrow[d, "\otimes_C"'] & D \times D \arrow[r, "\otimes_D"] & D \\ C \arrow[urr, dotted, "F \otimes G"'] & & \end{tikzcd}$$

Spectral Green functors

In our situation, take product on $\mathrm{Span}(\mathbf{F}_G)$ induced by the cartesian product of finite G-sets, and smash product on $\mathbf{Sp}$. Obtains symmetric monoidal structure on $\mathbf{Sp}_G$. Spectral Green functors are then commutative algebra objects in $\mathbf{Sp}_G$.

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Every object in $\mathrm{Span}(\mathbf{F}_G)$ is a commutative algebra via $c \times c \xleftarrow{\Delta} c \xrightarrow{=} c$, so the values of a spectral Green functor are commutative algebras in $\mathbf{Sp}$. More work: contravariant functoriality yields algebra maps, covariant yields module maps.

HHR norm

Equivariantly, we want not only to discuss ordinary smash products, but also smash products where the indexing set admits a group action. For this, need to interpolate between $\mathbf{Sp}_H$ ranging over subgroups $H \leq G$.
Additively, have induction $\mathrm{Ind}_H^G : \mathbf{Sp}_H \longrightarrow \mathbf{Sp}_G$ and multiplicatively, have HHR norm $N_H^G : \mathbf{Sp}_H \longrightarrow \mathbf{Sp}_G$.

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Use coinduction functors in cartesian situation (i.e. spaces):
$\mathrm{Coind}_H^G = N_H^G : \mathbf{Top}_H \longrightarrow \mathbf{Top}_G$ right adjoint to Res_H^G, $X \mapsto \mathrm{Map}_H(G, X)$.

Pointing: Define symmetric monoidal functors (with respect to smash product) $N_H^G : \mathbf{Top}_{H,*} \longrightarrow \mathbf{Top}_{G,*}$ which prolong $\mathrm{Coind}_H^G : \mathbf{Top}_H \longrightarrow \mathbf{Top}_G$ as a symmetric monoidal left Kan extension.

$$\begin{tikzcd} \mathbf{Top}_H \arrow[r, "\mathrm{Coind}_H^G"] \arrow[d, "(-)_+"] & \mathbf{Top}_G \arrow[d, "(-)_+"] \\ \mathbf{Top}_{H,*} \arrow[r, dotted, "N_H^G"] & \mathbf{Top}_{G,*} \end{tikzcd}$$

HHR norm

Explicitly, for X a pointed space, $N_1^G(X)(G/H) \simeq X^{\wedge[G:H]}$. Observe then that $\mathrm{N}_1^G(S^1) \simeq S^\rho$, so the regular representation sphere appears from purely categorical considerations.

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Ditto for defining $N_H^G : \mathbf{Sp}_H \longrightarrow \mathbf{Sp}_G$:

$$\begin{tikzcd}
\mathbf{Top}_{H,*} \arrow[r, "N_H^G"] \arrow[d, "\Sigma^\infty"] & \mathbf{Top}_{G,*} \arrow[d, "\Sigma^\infty"] \\
\mathbf{Sp}_H \arrow[r, dotted, "N_H^G"] & \mathbf{Sp}_G
\end{tikzcd}$$

Facts: $\mathrm{Res}^G N^G \simeq (-)^{\otimes |G|}$ and $\Phi^G N^G \simeq id$. $(N^G X)^G$ generally hard to understand.

To compute N^G, can write a spectrum as a filtered colimit of desuspensions of suspension spectra.

G-symmetric monoidal structures

The collection $\{\mathbf{Sp}_H : H \leq G\}$ of symmetric monoidal ∞-categories together with the norm functors interpolating between them cohere into a more sophisticated 'G-symmetric monoidal' structure.

Recall: To endow an ∞-category C with the added *structure* of a symmetric monoidal ∞-category is to define a cocartesian fibration $C^{\otimes} \longrightarrow \mathbf{F}_*$, satisfying a Segal condition that decomposes $(C^{\otimes})_{n_+} \simeq C^{\times n}$.

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Rough idea: Basic objects are G-categories i.e. cocartesian fibrations over $\mathbf{O}_G^{op}$. A G-SMC is a cocartesian fibration $\pi : C^{\otimes} \longrightarrow \underline{\mathbf{F}}_{G,*}$ down to the G-category of finite pointed G-sets, whose fiber over G/H is the category of finite pointed H-sets: $(\underline{\mathbf{F}}_{G,*})_{G/H} \simeq \mathbf{F}_{H,*}$. π is required to satisfy an appropriate G-Segal condition. Norm functors N_H^G are encoded by collapse maps $G/H \longrightarrow G/G$, whereas the ordinary smash product is recorded by fold maps $G/H \sqcup G/H \longrightarrow G/H$.

G-commutative algebras

Recall: given a symmetric monoidal ∞-category $\pi : C^{\otimes} \longrightarrow \mathbf{F}_*$, a commutative algebra object A in C is a section $\sigma : \mathbf{F}_* \longrightarrow C^{\otimes}$ of π, with $\sigma(1_+) = A$, which sends "inert" maps to cocartesian edges: the effect being that $\sigma(n_+) \simeq (A, ..., A)$ and the unique active map $n_+ \longrightarrow 1_+$ that points $n \longrightarrow 1$ induces the multiplication $A \otimes ... \otimes A \longrightarrow A$.

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Similarly, we define a G-commutative algebra object A in a G-SMC C to be a suitable section of $C^{\otimes} \longrightarrow \underline{\mathbf{F}}_{G,*}$. Get maps $N_H^G \operatorname{Res}_H^G A \longrightarrow A$ in addition to usual $A \otimes A \longrightarrow A$ (identifying A with object in fiber $C_{G/G}$).

In particular, we can norm elements in homotopy of G-commutative ring spectrum: $x \in \pi_n(\operatorname{Res}^G A) \Rightarrow N^G(x) \in \pi_{\rho n}(A)$.

Remark: levelwise π_0 of a G-commutative ring spectrum obtains a Tambara functor (Ullman's theorem).

Examples of G-commutative algebras

Principle (Barwick, Gepner-Groth-Nikolaus, May in older language, ...): K-theory (group completion or something more sophisticated) is lax symmetric monoidal. Ergo, equivariant K-theory transforms categorical Green functors into spectral Green functors.

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In framework of G-symmetric monoidal categories, can further show that equivariant K-theory is lax G-symmetric monoidal (work in progress by Barwick, Guillou-May-Merling (?)). For example, obtain G-commutative algebra structure on KU_G, using the additional information of the indexed tensor products (of course, this example is already known).

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Another example: Galois equivariant K-theory. Let E/k be a finite Galois extension with Galois group G, and define Mackey functor $G/H \mapsto \mathrm{Vect}(E^H)$. Get norm $N_1^G : \mathrm{Vect}(E) \longrightarrow \mathrm{Vect}(k)$ by using descent data on $V^{\otimes G}$.

Beyond equivariant homotopy theory

To make sense of the preceding ideas, we used more the orbit category of G rather than G itself. In particular, nothing was reliant on the representation theory of G (even the regular representation sphere S^ρ was defined abstractly). We can play the same game with other categories $\mathbf{O}$ that resemble $\mathbf{O}_G$, in the sense that:

1. $\mathbf{O}$ is *orbital*: the finite coproduct completion $\mathbf{F} = \mathbf{O}^{\sqcup}$ admits pullbacks, so that the span construction is sensible.
2. $\mathbf{O}$ is *atomic*: $\mathbf{O}$ admits no-nontrivial retracts. Technical hypothesis needed for the stability theory to work.

Examples: $\mathbf{O} = \mathbf{F}^{\mathrm{surj}}_{\leq n} \rightsquigarrow n$-excisive functors $\mathbf{Sp} \longrightarrow \mathbf{Sp}$ (Glasman), cyclonic orbit categories of Barwick/Glasman $\rightsquigarrow S^1$-spectra genuine relative to finite subgroups, global equivariant spectra with respect to finite groups, ...

Non-example: G-spectra for compact Lie group G of dimension > 0 (because of presence of dimension shifting transfers).

Thanks for listening!