Parametrized higher category theory

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Answer: depends on the class of weak equivalences one inverts in the larger category of G-spaces.

Inverting the class of maps that induce a weak equivalence of underlying spaces, $X \rightsquigarrow$ the homotopy type of the underlying space X, together with the homotopy coherent G-action. Can extract homotopy fixed points and orbits X^{hG}, X_{hG} from this.

Elmendorf's theorem

But we might also want to retain the data of the actual fixed point spaces X^H. To do this, we can instead invert the smaller class of maps that induce a weak equivalence on all fixed points. Then the resulting "homotopy type" of X knows the homotopy types of all the X^H, together with the restriction and conjugation maps relating them.

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Elmendorf's theorem: $\mathrm{Top}_G[\mathscr{W}^{-1}] \simeq \mathrm{Fun}(\mathbf{O}_G^{op}, \mathbf{Top})$ where Top_G is a category of (nice) topological spaces with G-action, $\mathscr{W}$ is the class of maps as above, **Top** is the ∞-category of spaces, and $\mathbf{O}_G$ is the orbit category of G.

Definition

$\mathbf{Top}_G := \mathrm{Fun}(\mathbf{O}_G^{op}, \mathbf{Top})$ is the ∞-category of G-spaces.

G-coproducts

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Exercise: Show that $\coprod_{G/H} X$ homeomorphic as a G-space to $\mathrm{Ind}_H^G \mathrm{Res}_H^G X$ for adjunction

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Suggests an answer, but only if we retain whole $\mathbf{O}_G$-*presheaf* of ∞-categories $(G/H) \mapsto \mathbf{Top}_H$.

Definition

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3. The ∞-category of *'genuine' G-spectra* $\mathbf{Sp}^G$, i.e. spectral Mackey functors on $\mathbf{F}_G$.
   Let $A^{eff}(\mathbf{F}_G)$ be the effective Burnside $(2,1)$-category of G, given by taking as objects finite G-sets, as morphisms spans of finite G-sets, and as 2-morphisms isomorphisms between spans. Then $\mathbf{Sp}^G := \mathrm{Fun}^{\oplus}(A^{eff}(\mathbf{F}_G), \mathbf{Sp})$, the ∞-category of direct-sum preserving functors from $A^{eff}(\mathbf{F}_G)$ to $\mathbf{Sp}$.

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The last option produces *transfer* maps, encoded by the covariant maps in $A^{eff}(\mathbf{F}_G)$ - ubiquitous in examples (e.g. K-theory).

Universal properties for G-spaces and G-spectra

Idea: G-spectra are obtained from G-spaces by stabilizing and enforcing the coincidence of G-coproducts and G-products (the Wirthmuller isomorphism $\mathrm{Ind}_H^G \simeq \mathrm{Coind}_H^G$).

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We saw in Denis' talk a theorem characterizing G-spectra along these lines. We will aim for a somewhat more formal counterpart concerning G-spaces.

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We will always work within the framework of quasi-categories (Boardman-Vogt, Joyal, Lurie).

∞-categories

Definition

An ∞-category C is a simplicial set which has inner horn fillers: for all $0 < k < n$ and maps $f : \Lambda^n_k \longrightarrow C$, f admits a (not necessarily unique!) extension $\overline{f} : \Delta^n \longrightarrow C$.

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E.g. $k = 1$, $n = 2$:

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This defines a weak composition law: if C is an ∞-category, then the restriction functor $\mathrm{Fun}(\Delta^2, C) \longrightarrow \mathrm{Fun}(\Lambda^2_1, C)$ is a trivial Kan fibration, so the filler is unique up to contractible choice.

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Higher fillers encode associativity coherences satisfied by the composition law.

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Mapping spaces: For objects $x, y \in C$, have Kan complex

$$\mathrm{Map}_C(x, y) := \{x\} \times_C \mathrm{Fun}(\Delta^1, C) \times_C \{y\}.$$

Can extract $\circ : \mathrm{Map}_C(x, y) \times \mathrm{Map}_C(y, z) \longrightarrow \mathrm{Map}_C(x, z)$ as above.

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Equivalences within an ∞-category C: edges $e : x \to y$ s.t. $e^* : \mathrm{Map}_C(y, x) \longrightarrow \mathrm{Map}_C(x, z)$ equivalence for all $z \in C$.

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Categorical equivalences: functors $F : C \longrightarrow D$ which are fully faithful and essentially surjective.

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E.g. let $\mathbf{Top}$ = homotopy coherent nerve of the simplical category of Kan complexes. Then

$$\begin{aligned} \mathbf{Top}_* &:= \mathbf{Top}_{(*)/} \\ \mathbf{Sp} &:= \mathrm{Exc}_*(\mathbf{Top}_*^{\mathrm{fin}}, \mathbf{Top}) \\ \mathbf{Sp}^G &:= \mathrm{Fun}^{\oplus}(A^{\mathrm{eff}}(\mathbf{F}_G), \mathbf{Sp}) \end{aligned}$$

∞-categories: join construction

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E.g. C, D ∞-categories $\Rightarrow$ join $C \star D$. Has $C, D \subset C \star D$ as full subcategories, no new objects, and $\mathrm{Map}_{C \star D}(c, d) = *$, $\mathrm{Map}_{C \star D}(d, c) = \emptyset$.

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Let $i : \partial\Delta^1 \longrightarrow \Delta^1$. Then obtain adjunction

$$i^* : s\mathbf{Set}_{/\Delta^1} \rightleftarrows s\mathbf{Set}_{/\partial\Delta^1} : i_*$$

and can define $C \star D := i_*(C, D)$. Explicitly, have n-simplices

$$\mathrm{Hom}_{/\Delta^1}(\Delta^n, C \star D) \cong \mathrm{Hom}(\Delta^{n_0}, C) \times \mathrm{Hom}(\Delta^{n_1}, D)$$

ranging over maps $\Delta^n \cong \Delta^{n_0} \star \Delta^{n_1} \longrightarrow \Delta^1$.

Get cone $K^{\triangleleft} = \Delta^0 \star K$, cocone $K^{\triangleright} = K \star \Delta^0$.

∞-categorical theory of (homotopy) colimits

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Definition

1. An object $x \in C$ is *initial* if for all $y \in C$, $\mathrm{Map}_C(x, y) \simeq *$.
2. Given a functor $p : K \longrightarrow C$, an extension $\overline{p} : K^{\triangleright} \longrightarrow C$ is a *colimit diagram* (and $\overline{p}$(cone pt) is a colimit of p) if in

$$\begin{tikzcd} \Delta^0 \arrow[r, "\{\overline{p}\}"] \arrow[dr, "="'] & C^{p/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(K \star \Delta^0, C) \arrow[d] \\ & \Delta^0 \arrow[r, "\{p\}"] & \mathrm{Fun}(K, C) \end{tikzcd}$$

$\{\overline{p}\}$ is an initial object, where $C^{p/}$ is defined as the pullback.

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Local $\Rightarrow$ global:

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{colim}}(K \star \Delta^0, C) \arrow[r, hook, "\subset"] \arrow[d, two heads, "\simeq"] & \mathrm{Fun}(K \star \Delta^0, C) \arrow[dl] \arrow[dr, shift left, "\mathrm{ev}_{\mathrm{cone}}"] & \\
\mathrm{Fun}(K, C) \arrow[u, dotted, bend left] & & C \arrow[ul, shift left, "\delta"]
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Fix an ∞-category S. Will work with functors $S \longrightarrow \mathbf{Cat}_\infty \Leftrightarrow$ cocartesian fibrations $C \longrightarrow S$ ($= S$-category).

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Natural transformation of functors $\Leftrightarrow$ map $C \longrightarrow D$ of simplicial sets over S which preserves cocartesian edges.

The necessity of remembering data of cocartesian edges leads one to consider *marked simplicial sets* (X, E), $E \subset X_1$ that contains the degenerate edges. Then cocartesian fibrations over S are the fibrant objects in a model structure on $s\mathbf{Set}^+_{/(S,S_1)}$.

Example: S-category of S-objects

The lax colimit functor $(C \longrightarrow S) \mapsto C$, $\mathbf{Cat}_{\infty/S}^{cocart} \longrightarrow \mathbf{Cat}_\infty$ admits a right adjoint $E \mapsto \underline{E}_S$, which sends E to the S-category of S-objects in E.

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To describe $\underline{E}_S$, note $(\underline{E}_S)_s \simeq \mathrm{Fun}_S(S^{s/}, E) \simeq \mathrm{Fun}(S^{s/}, E)$, and the functoriality given by that in $S^{(-)/}$.

E.g. if $S = \mathbf{O}_G^{op}$, then $\underline{\mathbf{Top}}_S = \underline{\mathbf{Top}}_G$, in view of the equivalence of categories $\mathbf{O}_H \simeq (\mathbf{O}_G)^{/(G/H)}$ (implemented by the induction functor).

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We will endow $\underline{\mathbf{Top}}_S$ with an additional universal mapping property for suitable functors *out* of $\underline{\mathbf{Top}}_S$.

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∞-category C	S-category $C \; (\longrightarrow S)$
Functor $C \longrightarrow D$	S-functor $C \longrightarrow D$
Functor ∞-category $\mathrm{Fun}(C, D)$	S-functor category $\underline{\mathrm{Fun}}_S(C, D)$
Join $C \star D$	S-join $C \star_S D$
Slice $C^{p/}$	S-slice $C^{(p,S)/}$
Initial object $x \in C$	S-initial object $\sigma : S \longrightarrow C$
Colimit diagram $\overline{p} : K \star \Delta^0 \longrightarrow C$	S-colimit diagram $\overline{p} : K \star_S S \longrightarrow C$
Adjunction $\mathrm{Fun}(K, C) \rightleftarrows C$	S-adjunction $\underline{\mathrm{Fun}}_S(K, C) \rightleftarrows C$

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E.g. $S = \mathbf{O}_G^{op}$, $\underline{\mathrm{Fun}}_G(C, D) : G/H \mapsto \mathrm{Fun}_H(\mathrm{Res}^G_H C, \mathrm{Res}^G_H D)$, functoriality given by restriction and conjugation.

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- $\sigma : S \longrightarrow C$ S-initial object $\Leftrightarrow$ $\sigma(s)$ initial $\forall s \in S$.

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- $C \star_S D : s \mapsto C_s \star D_s$ fiberwise join.
- $\sigma : S \longrightarrow C$ S-initial object $\Leftrightarrow$ $\sigma(s)$ initial $\forall s \in S$.
- $\overline{p} : K \star_S S \longrightarrow C$ S-colimit diagram $\Leftrightarrow$ $\sigma_{\overline{p}}$ S-initial object

$$\begin{tikzcd}
S \arrow[r, "\sigma_{\overline{p}}"] \arrow[dr, "="'] & C^{(p,S)/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Fun}}_S(K \star_S S, C) \arrow[d] \\
& S \arrow[r, "\sigma_p"] & \underline{\mathrm{Fun}}_S(K, C)
\end{tikzcd}$$

Absolute $\Rightarrow$ Parametrized

- Internal hom $\underline{\mathrm{Fun}}_S(C, D)$ in $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$:

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Base-change subtlety: $\overline{p}$ not just initial cocartesian extension, but pullback via every $S^{s/} \longrightarrow S$ is as well ("G-colimit restricts to H-colimit for all subgroups H in G").

Example of S-colimits

- Fiberwise colimits: Suppose $K = S \times L$ constant diagram. Then a S-colimit of $p : S \times L \longrightarrow C$ exists if $\forall s \in S$, C_s admits colimit of $p_s : L \longrightarrow C_s$, and $\forall (\alpha : s \to t) \in S$, $\alpha_! : C_s \longrightarrow C_t$ preserves colimit of p_s.

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- S-coproducts: Pull back to $S^{s/}$ and suppose $K = \coprod_i S^{\alpha_i/} \simeq \coprod_i S^{t_i/} \longrightarrow S^{s/}$ for a collection of morphisms $\{\alpha_i : s \to t_i\}$, i.e. K is a disjoint union of corepresentable fibrations over $S^{s/}$. Then a $S^{s/}$-colimit for some $p : K \longrightarrow C \times_S S^{s/}$ is, in particular, the disjoint union of $\mathrm{Ind}_{t_i}^{s}(p(t_i))$ in C_s when the left adjoints $\mathrm{Ind}_{t_i}^{s} : C_{t_i} \longrightarrow C_s$ to $\alpha_{i!}$ exist. (This ignores the base-change condition.)

S-coproducts, continued

Let $T = S^{op}$ and define a cartesian fibration $C^{\times} \longrightarrow \mathbf{F}_T$ by $U = \coprod_i s_i \mapsto \mathrm{Fun}_S(\underline{U}, C) \simeq \prod_i C_{s_i}$.

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Proposition

Suppose T is orbital, i.e. $\mathbf{F}_T$ admits pullbacks. Then C admits finite S-coproducts if and only if $\pi : C^{\times} \longrightarrow \mathbf{F}_T$ is a **Beck-Chevalley fibration***, i.e. π is both cocartesian and cartesian, and for every pullback square*

$$\begin{tikzcd} W \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] \arrow[dr, phantom, "\lrcorner", very near start] & V' \arrow[d, "\beta"] \\ U \arrow[r, "\alpha"] & V \end{tikzcd}$$

in $\mathbf{F}_T$, the natural transformation

$$(\alpha')_!(\beta')^* \longrightarrow \beta^* \alpha_! \tag{0.5.1}$$

adjoint to the equivalence $(\beta')^*\alpha^* \simeq (\alpha')^*\beta^*$ is itself an equivalence.

S-colimits in S-categories of S-objects

Given $p : K \longrightarrow \underline{E}_S$, let $p^\dagger : K \longrightarrow E$ denote the corresponding functor under the equivalence $\mathrm{Fun}_S(K, \underline{E}_S) \simeq \mathrm{Fun}(K, E)$. Then $\overline{p}|_S : S \longrightarrow \underline{E}_S$ is a S-colimit of p if and only if

$$\begin{tikzcd} K \arrow[r, "p^\dagger"] \arrow[d] & E \\ S \arrow[ur, "\overline{p}^\dagger|_S"'] & \end{tikzcd}$$

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Since left Kan extensions are computed as fiberwise colimits, the base-change condition is automatic.

Remark: For $S = \mathbf{O}_G^{op}$, this is situation of Dotto–Moi (under Elmendorf theorem type correspondence).

Passage to slices

"Classical" indexed category theory deals with the situation where the base has an initial object e.g. $S = \mathbf{O}_G^{op}$ but not S any Kan complex. What does this additional generality buy us?

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Answer: $p : K \longrightarrow C$ admits an extension to an S-colimit diagram $\overline{p} : K \star_S S \longrightarrow C$ if and only if $\forall s \in S$, the pullback $p_{\underline{s}} : K \times_S S^{s/} \longrightarrow C \times_S S^{s/}$ admits an extension to an $S^{s/}$-colimit diagram.

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If S is a connected Kan complex i.e. $S \simeq BX$, then we recover the familiar fact that colimits in $\mathrm{Fun}(BX, \mathbf{Cat}_\infty)$ are created by the evaluation functor to $\mathbf{Cat}_\infty$.

S-colimits more generally

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We don't yet know how to show that an S-category C even admits all S-colimits, unless it is a S-category of S-objects in a cocomplete ∞-category.

More precisely: say that an S-category C is *S-cocomplete* if $\forall s \in S$, $C \times_S S^{s/}$ admits all $S^{s/}$-colimits. We want some criteria for when C is S-cocomplete.

Proposition (S.)

Suppose S^{op} orbital. Then C is S-cocomplete if and only if it admits (constant) geometric realizations and S-coproducts.

Proof.

S-generalization of the Bousfield-Kan formula. $\square$

Bousfield-Kan formula

Let $p : K \longrightarrow C$ be a functor. Have formula for the colimit of p:

$$\left| \coprod_{x \in K_0} p(x) \leftleftarrows \coprod_{\alpha \in K_1} p(\alpha(0)) \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \coprod_{\sigma \in K_2} p(\sigma(0)) \;\; \ldots \right|$$

This is really a statement about K. Consider the span

$$\begin{tikzcd} K_n \arrow[r] \arrow[d] & \Delta^{op}_{/K} \arrow[r, "\upsilon_K"] \arrow[d, "\rho_K"] & K \\ \{[n]\} \arrow[r] & \Delta^{op} & \end{tikzcd}$$

where υ_K is the first vertex map. Then υ_K is cofinal and ρ is a cocartesian fibration.

Using essentially that an ∞-category is presented as a simplicial set!

S Bousfield-Kan formula

We can generalize the previous span to

$$\begin{tikzcd}
\coprod_{\sigma \in K_n} S^{\pi\sigma(n)/} \arrow[r, hook] \arrow[d] & \mathrm{Simp}_S(K) \arrow[r, "\nu_{K,S}"] \arrow[d, "\rho_{K,S}"] & K \arrow[d, "\pi"] \\
\{[n]\} \times S \arrow[r, hook] & \Delta^{op} \times S \arrow[r] & S
\end{tikzcd}$$

and show that $\nu_{K,S}$ is S-cofinal, $\rho_{K,S}$ is S-cocartesian.

S-colimits more generally

Usual case: For S-category C, fibers C_s are presentable, and pushforward ($=$ restriction) functors preserve limits and colimits. Then one just has to check the base-change ($=$ Mackey decomposition) condition for the left adjoints.

Example

The S-category of spectra $\underline{\mathbf{Sp}}^S : \{s\} \mapsto \mathbf{Mack}(\mathbf{F}_{(S^{s/})^{op}}, \mathbf{Sp})$ is S-cocomplete.

Computing S-colimits in S-spectra

We already know that the geometric fixed points functors preserves colimits, as a left adjoint. Would like to promote to a G-left adjoint, for it to preserve G-colimits.

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H-geometric fixed points a la Mackey functors: $\phi : \mathbf{O}_{G,H} \subset \mathbf{O}_G$ full subcategory on orbits s.t. H-acts trivially. Then the coproduct-preserving extension $\phi^{\sqcup} : \mathbf{F}_{G,H} \longrightarrow \mathbf{F}_G$ admits a right adjoint ϕ^{-1} that also preserves coproducts $\rightsquigarrow A^{eff}(\mathbf{F}_G) \longrightarrow A^{eff}(\mathbf{F}_{G,H})$. Get adjunction

$$\phi^* : \mathbf{Sp}^G \rightleftarrows \mathbf{Sp}^{G,H} : \phi_*$$

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where the left adjoint is H-geometric fixed points.
The full subcategory $\mathbf{O}^{op}_{G,H} \subset \mathbf{O}^{op}_G$ interacts nicely when pulled back along the slice categories of $\mathbf{O}^{op}_G$, so this gets promoted to a G-adjunction

$$\phi^* : \underline{\mathbf{Sp}}^G \rightleftarrows \underline{\mathbf{Sp}}^{G,H} : \phi_*.$$

Computing C_p-colimits in C_p-spectra

Example

Let $G = C_p, H = C_p$. Then we get

$$\begin{tikzcd} \mathbf{Sp}^{C_p} \arrow[r, shift left, "\Phi^{C_p}"] \arrow[d, "\mathrm{Res}^{C_p}_1"'] & \mathbf{Sp} \arrow[l, shift left, "R"] \arrow[d] \\ \mathbf{Sp} \arrow[r, shift left] & \ast. \arrow[l, shift left, "0"] \end{tikzcd}$$

Note that $R(X)$ is concentrated on C_p/C_p and $\Phi^{C_p} R \simeq id$.

A C_p-colimit of $K \longrightarrow \underline{\mathbf{Sp}}^{C_p, C_p}$ on the RHS reduces to the colimit of $K_{C_p/C_p} \longrightarrow \mathbf{Sp}$.

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Beware: certainly false for categorical fixed points $(-)^{C_p}$, even though this preserves (ordinary) colimits: $(\mathrm{Ind}_1^{C_p}(X))^{C_p} \simeq (\mathrm{Coind}_1^{C_p}(X))^{C_p} \simeq X$, not 0.

Example: Thom spectra

Let's build $MU_{\mathbf{R}}$ as a C_2-colimit. Want $\Phi^1(MU_{\mathbf{R}}) = MU$ with the complex conjugation C_2-action and $\Phi^{C_2}(MU_{\mathbf{R}}) = MO$. Look for:

$$\begin{tikzcd}
 & & & \mathrm{BGL}_1(\mathbf{S}) \arrow[d, hook] \\
\mathrm{BO} \arrow[urrr, bend left, "J_O"] \arrow[r, dotted, "J_U^{C_2}"'] \arrow[d] & \mathrm{BGL}_1(\mathbf{S}_{C_2}) \arrow[urr] \arrow[r, hook] \arrow[d] & \mathbf{Sp}^{C_2} \arrow[r, "\Phi^{C_2}"] \arrow[d, "\mathrm{Res}_1^{C_2}"] & \mathbf{Sp} \arrow[d] \\
\mathrm{BU} \arrow[r, "J_U"] & \mathrm{BGL}_1(\mathbf{S}) \arrow[r, hook] & \mathbf{Sp} \arrow[r] & *
\end{tikzcd}$$

$J_U^{C_2}$ obtained from colimit over C_2-maps $U(n) \longrightarrow \mathrm{Aut}_*(S^{\rho n}, S^{\rho n})$.

Vista: Understand how to produce G-Thom spectra (G-)categorically.

Example: G-cubes after Dotto-Moi

Let J be a finite G-set. G acts on $\mathcal{P}(J)$ by $U \mapsto g \cdot U$. Get cocartesian fibration $\underline{\mathcal{P}(J)} \longrightarrow \mathbf{O}_G^{op}$, $\underline{\mathcal{P}(J)}_{G/H} = \mathcal{P}(J)^H$.

Have two G-fixed points, $\emptyset$ and $J \rightsquigarrow$ two cocartesian sections of $\underline{\mathcal{P}(J)}$, and moreover decompositions $\underline{\mathcal{P}(J)} \cong \mathbf{O}_G^{op} \star_{\mathbf{O}_G^{op}} \underline{\mathcal{P}(J) \backslash \{\emptyset\}} \cong \underline{\mathcal{P}(J) \backslash \{J\}} \star_{\mathbf{O}_G^{op}} \mathbf{O}_G^{op}$ Thus, it is sensible to say that a G-functor $\underline{C} \longrightarrow \underline{D}$ is *J-excisive*: it sends G-colimit diagrams $\underline{\mathcal{P}(J)} \longrightarrow \underline{C}$ to G-limit diagrams $\underline{\mathcal{P}(J)} \longrightarrow \underline{D}$.

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Theorem (Dotto-Moi)

$\mathbf{Sp}^G \simeq \mathrm{Exc}_*^G(\underline{\mathbf{Top}}_{G,*}^{fin}, \underline{\mathbf{Top}}_G)$ *where the right-hand side is the full subcategory of* $\mathrm{Fun}_G(\underline{\mathbf{Top}}_{G,*}^{fin}, \underline{\mathbf{Top}}_G)$ *on the pointed G_+-excisive functors.*

S-Yoneda lemma

Naive and wrong idea: Embed C into a S-category $\mathbf{P}_S(C)$ s.t. on fibers, $C_s \mapsto \mathrm{Fun}(C_s^{op}, \mathbf{Top})$. One problem is that we don't have functoriality in the target.

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Vertical opposite: $C^{vop} \longrightarrow S$ is the cocartesian fibration $\Leftrightarrow s \mapsto C_s^{op}$.

Have S-mapping space functor $C^{vop} \times_S C \longrightarrow \underline{\mathbf{Top}}_S$, $(x, y) \mapsto \mathrm{Map}_{C_s}(x, y)$ (viewed as a functor to $\mathbf{Top}$). Adjoint yields S-Yoneda embedding

$$j : C \longrightarrow \mathbf{P}_S(C) := \underline{\mathrm{Fun}}_S(C^{vop}, \underline{\mathbf{Top}}_S).$$

Have factorization $j_s : C_s \longrightarrow \mathrm{Fun}(C_s^{op}, \mathbf{Top}) \longrightarrow \mathrm{Fun}(C^{vop} \times_S S^{s/}, \mathbf{Top})$.

S-Yoneda lemma

Let $\underline{\mathrm{Fun}}^L_S(C, D)$ be the full S-subcategory whose fiber over s consists of those functors that *strongly* preserve $S^{s/}$-colimits (= preserve $S^{t/}$-colimits after pullback by every $S^{t/} \longrightarrow S^{s/}$).

Theorem (S., Riehl–Verity)

$\mathbf{P}_S(C)$ is S-cocomplete, and for any S-cocomplete E, restriction along the S-Yoneda embedding yields an equivalence

$$\underline{\mathrm{Fun}}^L_S(\mathbf{P}_S(C), E) \longrightarrow \underline{\mathrm{Fun}}_S(C, E).$$

Proof.

Inverse given by left S-Kan extension along j. Need to identify LHS with those functors which are left S-Kan extensions of their restriction to C. $\square$

Letting $C = S = \mathbf{O}_G^{op}$, prove Hill's conjecture.

Thanks for listening!