Parametrized higher category theory

Jay Shah

MIT

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Answer: depends on the class of weak equivalences one inverts in the larger category of G-spaces.

Inverting the class of maps that induce a weak equivalence of underlying spaces, $X \rightarrow$ the homotopy type of the underlying space X, together with the homotopy coherent G-action. Can extract homotopy fixed points and orbits X^{hG} , X_{hG} from this.

But we might also want to retain the data of the actual fixed point spaces X^{H} . To do this, we can instead invert the smaller class of maps that induce a weak equivalence on all fixed points. Then the resulting "homotopy type" of X knows the homotopy types of all the X^{H} , together with the restriction and conjugation maps relating them.

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Elmendorf's theorem: $\operatorname{Top}_G[\mathscr{W}^{-1}] \simeq \operatorname{Fun}(\mathbf{O}_G^{op}, \mathbf{Top})$ where Top_G is a category of (nice) topological spaces with *G*-action, \mathscr{W} is the class of maps as above, **Top** is the ∞ -category of spaces, and \mathbf{O}_G is the orbit category of *G*.

Definition

 $\mathbf{Top}_{\mathcal{G}} := \mathsf{Fun}(\mathbf{O}_{\mathcal{G}}^{op}, \mathbf{Top})$ is the ∞ -category of \mathcal{G} -spaces.

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Exercise: Show that $\coprod_{G/H} X$ homeomorphic as a *G*-space to $\operatorname{Ind}_{H}^{G} \operatorname{Res}_{H}^{G} X$ for adjunction

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: $\operatorname{Top}_{H} \rightleftharpoons \operatorname{Top}_{G}$: $\operatorname{Res}_{H}^{G}$.

Suggests an answer, but only if we retain whole O_G -presheaf of ∞ -categories $(G/H) \mapsto Top_H$.

Definition

Top_{*G*} : $G/H \mapsto$ **Top**_{*H*} is the *G*-∞-category of *G*-spaces.

G-spectra

Let G be a finite group. We have three possibilities for a sensible notion of G-spectra:

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- 2 The ∞-category of 'naive' G-spectra, i.e. spectral presheaves on O_G. This is Fun(O^{op}_G, Sp), which is the stabilization of Top_G.

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 The ∞-category of 'genuine' G-spectra Sp^G, i.e. spectral Mackey functors on F_G. Let A^{eff}(F_G) be the effective Burnside (2,1)-category of G, given by taking as objects finite G-sets, as morphisms spans of finite G-sets, and as 2-morphisms isomorphisms between spans. Then Sp^G := Fun[⊕](A^{eff}(F_G), Sp), the ∞-category of direct-sum preserving functors from A^{eff}(F_G) to Sp. Let G be a finite group. We have three possibilities for a sensible notion of G-spectra:

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The last option produces *transfer* maps, encoded by the covariant maps in $A^{\text{eff}}(\mathbf{F}_{G})$ - ubiquitous in examples (e.g. *K*-theory).

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We saw in Denis' talk a theorem characterizing G-spectra along these lines. We will aim for a somewhat more formal counterpart concerning G-spaces. $\infty\text{-}\mathsf{category}=\mathsf{category}$ (weakly) enriched in $\infty\text{-}\mathsf{groupoids}$ (i.e. spaces, under homotopy hypothesis).

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We will always work within the framework of quasi-categories (Boardman-Vogt, Joyal, Lurie).

Definition

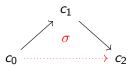
An ∞ -category C is a simplicial set which has inner horn fillers: for all 0 < k < n and maps $f : \Lambda_k^n \longrightarrow C$, f admits a (not necessarily unique!) extension $\overline{f} : \Delta^n \longrightarrow C$.

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Higher fillers encode associativity coherences satisfied by the composition law.

Mapping spaces: For objects $x, y \in C$, have Kan complex

$$\mathsf{Map}_{\mathcal{C}}(x,y) := \{x\} \times_{\mathcal{C}} \mathsf{Fun}(\Delta^{1},\mathcal{C}) \times_{\mathcal{C}} \{y\}.$$

Can extract \circ : $Map_C(x, y) \times Map_C(y, z) \longrightarrow Map_C(x, z)$ as above.

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Equivalences within an ∞ -category C: edges $e : x \to y$ s.t. $e^* : \operatorname{Map}_C(y, x) \longrightarrow \operatorname{Map}_C(x, z)$ equivalence for all $z \in C$.

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Categorical equivalences: functors $F : C \longrightarrow D$ which are fully faithful and essentially surjective.

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E.g. let $\mathbf{Top}=\mathsf{homotopy}$ coherent nerve of the simplical category of Kan complexes. Then

$$egin{aligned} \mathbf{Top}_* &\coloneqq \mathbf{Top}_{(*)/} \ \mathbf{Sp} &\coloneqq \mathsf{Exc}_*(\mathbf{Top}^{\mathsf{fin}}_*, \mathbf{Top}) \ \mathbf{Sp}^G &\coloneqq \mathsf{Fun}^\oplus(A^{\mathsf{eff}}(\mathbf{F}_G), \mathbf{Sp}) \end{aligned}$$

∞ -categories: join construction

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E.g. $C, D \propto$ -categories \Rightarrow join $C \star D$. Has $C, D \subset C \star D$ as full subcategories, no new objects, and $\operatorname{Map}_{C\star D}(c, d) = *$, $\operatorname{Map}_{C\star D}(d, c) = \emptyset$.

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Let $i : \partial \Delta^1 \longrightarrow \Delta^1$. Then obtain adjunction

$$i^* \colon s\mathbf{Set}_{/\Delta^1} \longleftrightarrow s\mathbf{Set}_{/\partial\Delta^1} : i_*$$

and can define $C \star D := i_*(C, D)$. Explicitly, have *n*-simplices

 $\operatorname{Hom}_{/\Delta^{1}}(\Delta^{n}, C \star D) \cong \operatorname{Hom}(\Delta^{n_{0}}, C) \times \operatorname{Hom}(\Delta^{n_{1}}, D)$ ranging over maps $\Delta^{n} \cong \Delta^{n_{0}} \star \Delta^{n_{1}} \longrightarrow \Delta^{1}$.

Get cone $K^{\triangleleft} = \Delta^0 \star K$, cocone $K^{\triangleright} = K \star \Delta^0$.

∞ -categorical theory of (homotopy) colimits

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Definition

- An object $x \in C$ is *initial* if for all $y \in C$, $Map_C(x, y) \simeq *$.
- Given a functor p: K → C, an extension p̄: K[▷] → C is a colimit diagram (and p̄(cone pt) is a colimit of p) if in

 $\{\overline{p}\}\$ is an initial object, where $C^{p/}$ is defined as the pullback.

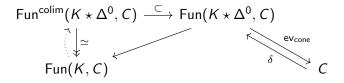
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Local \Rightarrow global:



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• $\alpha : s \longrightarrow t \in S, x \in C_s \Rightarrow \text{Obtain lift } \widetilde{\alpha} : x \longrightarrow y \text{ of } \alpha, \text{ where } y ≃ \alpha_!(x) \text{ for } \alpha_! : C_s \longrightarrow C_t \text{ the corresponding pushforward functor. These lifts are called$ *cocartesian edges*.

Natural transformation of functors \Leftrightarrow map $C \longrightarrow D$ of simplicial sets over S which preserves cocartesian edges.

The necessity of remembering data of cocartesian edges leads one to consider *marked simplicial sets* (X, E), $E \subset X_1$ that contains the degenerate edges. Then cocartesian fibrations over S are the fibrant objects in a model structure on $s\mathbf{Set}^+_{/(S,S_1)}$.

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To describe \underline{E}_S , note $(\underline{E}_S)_s \simeq \operatorname{Fun}_S(S^{s/}, E) \simeq \operatorname{Fun}(S^{s/}, E)$, and the functoriality given by that in $S^{(-)/}$.

E.g. if $S = \mathbf{O}_{G}^{op}$, then $\underline{\mathbf{Top}}_{S} = \underline{\mathbf{Top}}_{G}$, in view of the equivalence of categories $\mathbf{O}_{H} \simeq (\mathbf{O}_{G})^{\overline{/(G/H)}}$ (implemented by the induction functor).

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We will endow $\underline{\mathbf{Top}}_{S}$ with an additional universal mapping property for suitable functors *out* of $\underline{\mathbf{Top}}_{S}$.

Riehl-Verity: Do ∞ -category theory within a suitable (∞ , 2)-category, e.g. deriving from a model category enriched over *s***Set**_{Joyal}.

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Absolute	Parametrized
∞ -category C	S-category $C (\longrightarrow S)$
Functor $C \longrightarrow D$	S-functor $C \longrightarrow D$
Functor ∞ -category Fun(C,D)	S-functor category $\underline{Fun}_{S}(C, D)$
Join $C \star D$	S-join C ★ _S D
Slice $C^{p/}$	S-slice $C^{(p,S)/}$
Initial object $x \in C$	S-initial object $\sigma: S \longrightarrow C$
Colimit diagram $\overline{p}: K \star \Delta^0 \longrightarrow C$	S-colimit diagram $\overline{p}: K \star_S S \longrightarrow C$
Adjunction $\operatorname{Fun}(K, C) \rightleftharpoons C$	S-adjunction $\underline{Fun}_{\mathcal{S}}(\mathcal{K},\mathcal{C}) \longleftrightarrow \mathcal{C}$

• Internal hom $\underline{\operatorname{Fun}}_{S}(C, D)$ in $\operatorname{Fun}(S, \operatorname{Cat}_{\infty})$:

$$\{s\} \mapsto \operatorname{Fun}_{S^{s/}}(C \times_S S^{s/}, D \times_S S^{s/})$$

E.g. $S = \mathbf{0}_{G}^{op}$, $\underline{\operatorname{Fun}}_{G}(C, D) : G/H \mapsto \operatorname{Fun}_{H}(\operatorname{Res}_{H}^{G}C, \operatorname{Res}_{H}^{G}D)$, functoriality given by restriction and conjugation.

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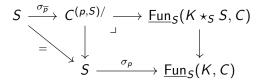
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Base-change subtlety: \overline{p} not just initial cocartesian extension, but pullback via every $S^{s/} \longrightarrow S$ is as well ("G-colimit restricts to H-colimit for all subgroups H in G").

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Example of S-colimits

Fiberwise colimits: Suppose K = S × L constant diagram. Then a S-colimit of p : S × L → C exists if ∀s ∈ S, C_s admits colimit of p_s : L → C_s, and ∀(α : s → t) ∈ S, α₁ : C_s → C_t preserves colimit of p_s.

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- S-coproducts: Pull back to $S^{s/}$ and suppose $K = \coprod_i S^{\alpha_i/} \simeq \coprod_i S^{t_i/} \longrightarrow S^{s/}$ for a collection of morphisms $\{\alpha_i : s \to t_i\}$, i.e. K is a disjoint union of corepresentable fibrations over $S^{s/}$. Then a $S^{s/}$ -colimit for some $p : K \longrightarrow C \times_S S^{s/}$ is, in particular, the disjoint union of $\operatorname{Ind}_{t_i}^s(p(t_i))$ in C_s when the left adjoints $\operatorname{Ind}_{t_i}^s : C_{t_i} \longrightarrow C_s$ to $\alpha_{i!}$ exist. (This ignores the base-change condition.)

S-coproducts, continued

Let $T = S^{op}$ and define a cartesian fibration $C^{\times} \longrightarrow \mathbf{F}_{T}$ by $U = \coprod_{i} s_{i} \mapsto \operatorname{Fun}_{S}(\underline{U}, C) \simeq \prod_{i} C_{s_{i}}.$

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Proposition

Suppose T is orbital, i.e. \mathbf{F}_T admits pullbacks. Then C admits finite S-coproducts if and only if $\pi : C^{\times} \longrightarrow \mathbf{F}_T$ is a **Beck-Chevalley fibration**, i.e. π is both cocartesian and cartesian, and for every pullback square

$$\begin{array}{ccc} W \xrightarrow{\alpha'} & V' \\ \downarrow^{\beta'} & \downarrow^{\beta} \\ U \xrightarrow{\alpha} & V \end{array}$$

in \mathbf{F}_{T} , the natural transformation

$$(\alpha')_{!}(\beta')^{*} \longrightarrow \beta^{*} \alpha_{!} \tag{0.5.1}$$

adjoint to the equivalence $(\beta')^* \alpha^* \simeq (\alpha')^* \beta^*$ is itself an equivalence.

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Parametrized higher category theory

Given $p: K \longrightarrow \underline{E}_S$, let $p^{\dagger}: K \longrightarrow E$ denote the corresponding functor under the equivalence $\operatorname{Fun}_S(K, \underline{E}_S) \simeq \operatorname{Fun}(K, E)$. Then $\overline{p}|_S: S \longrightarrow \underline{E}_S$ is a S-colimit of p if and only if



is a left Kan extension of p^{\dagger} (computed as the fiberwise colimit).

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Remark: For $S = \mathbf{O}_{G}^{op}$, this is situation of Dotto-Moi (under Elmendorf theorem type correspondence).

Jay Shah (MIT)

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"Classical" indexed category theory deals with the situation where the base has an initial object e.g. $S = \mathbf{O}_{G}^{op}$ but not S any Kan complex. What does this additional generality buy us?

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Answer: $p: K \longrightarrow C$ admits an extension to an *S*-colimit diagram $\overline{p}: K \star_S S \longrightarrow C$ if and only if $\forall s \in S$, the pullback $p_{\underline{s}}: K \times_S S^{s/} \longrightarrow C \times_S S^{s/}$ admits an extension to an $S^{s/}$ -colimit diagram. "Classical" indexed category theory deals with the situation where the base has an initial object e.g. $S = \mathbf{0}_{G}^{op}$ but not S any Kan complex. What does this additional generality buy us?

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If S is a connected Kan complex i.e. $S \simeq BX$, then we recover the familiar fact that colimits in Fun (BX, Cat_{∞}) are created by the evaluation functor to Cat_{∞} .

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More precisely: say that an S-category C is S-cocomplete if $\forall s \in S$, $C \times_S S^{s/}$ admits all $S^{s/}$ -colimits. We want some criteria for when C is S-cocomplete.

Proposition (S.)

Suppose S^{op} orbital. Then C is S-cocomplete if and only if it admits (constant) geometric realizations and S-coproducts.

Proof.

S-generalization of the Bousfield-Kan formula.

Jay Shah (MIT)

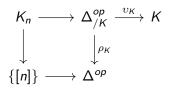
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Let $p: K \longrightarrow C$ be a functor. Have formula for the colimit of p:

$$\bigsqcup_{x \in K_0} p(x) := \bigsqcup_{\alpha \in K_1} p(\alpha(0)) := \bigsqcup_{\sigma \in K_2} p(\sigma(0)) \ldots$$

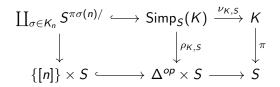
This is really a statement about K. Consider the span



where $v_{\mathcal{K}}$ is the first vertex map. Then $v_{\mathcal{K}}$ is cofinal and ρ is a cocartesian fibration.

Using essentially that an ∞ -category is presented as a simplicial set!

We can generalize the previous span to



and show that $\nu_{K,S}$ is S-cofinal, $\rho_{K,S}$ is S-cocartesian.

Usual case: For S-category C, fibers C_s are presentable, and pushforward (= restriction) functors preserve limits and colimits. Then one just has to check the base-change (= Mackey decomposition) condition for the left adjoints.

Example

The *S*-category of spectra \underline{Sp}^{S} : $\{s\} \mapsto Mack(F_{(S^{s/})^{op}}, Sp)$ is *S*-cocomplete.

Computing S-colimits in S-spectra

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H-geometric fixed points a la Mackey functors: $\phi : \mathbf{O}_{G,H} \subset \mathbf{O}_G$ full subcategory on orbits s.t. *H*-acts trivially. Then the coproduct-preserving extension $\phi^{\sqcup} : \mathbf{F}_{G,H} \longrightarrow \mathbf{F}_G$ admits a right adjoint ϕ^{-1} that also preserves coproducts $\rightsquigarrow A^{\text{eff}}(\mathbf{F}_G) \longrightarrow A^{\text{eff}}(\mathbf{F}_{G,H})$. Get adjunction

$$\phi^* \colon \mathbf{Sp}^{\mathbf{G}} \longleftrightarrow \mathbf{Sp}^{\mathbf{G},H} : \phi_*$$

where the left adjoint is *H*-geometric fixed points.

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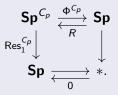
where the left adjoint is *H*-geometric fixed points. The full subcategory $\mathbf{O}_{G,H}^{op} \subset \mathbf{O}_{G}^{op}$ interacts nicely when pulled back along the slice categories of \mathbf{O}_{G}^{op} , so this gets promoted to a *G*-adjunction

$$\phi^* \colon \underline{\mathbf{Sp}}^{G} \longleftrightarrow \underline{\mathbf{Sp}}^{G,H} : \phi_*.$$

Computing C_p -colimits in C_p -spectra

Example

Let $G = C_p$, $H = C_p$. Then we get



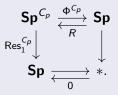
Note that R(X) is concentrated on C_p/C_p and $\Phi^{C_p}R \simeq id$.

A C_p -colimit of $K \longrightarrow \underline{Sp}^{C_p, C_p}$ on the RHS reduces to the colimit of $K_{C_p/C_p} \longrightarrow Sp$.

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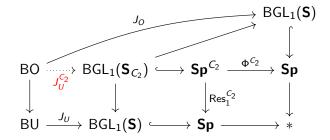


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A C_p -colimit of $K \longrightarrow \underline{Sp}^{C_p, C_p}$ on the RHS reduces to the colimit of $\mathcal{K}_{C_p/C_p} \longrightarrow \mathbf{Sp}$. Beware: certainly false for categorical fixed points $(-)^{C_p}$, even though this preserves (ordinary) colimits: $(\operatorname{Ind}_1^{C_p}(X))^{C_p} \simeq (\operatorname{Coind}_1^{C_p}(X))^{C_p} \simeq X$, not 0.

Example: Thom spectra

Let's build MU_R as a C_2 -colimit. Want $\Phi^1(MU_R) = MU$ with the complex conjugation C_2 -action and $\Phi^{C_2}(MU_R) = MO$. Look for:



 $J_U^{C_2}$ obtained from colimit over C_2 -maps $U(n) \longrightarrow \operatorname{Aut}_*(S^{\rho n}, S^{\rho n})$.

Vista: Understand how to produce G-Thom spectra (G-)categorically.

Let J be a finite G-set. G acts on $\mathcal{P}(J)$ by $U \mapsto g \cdot U$. Get cocartesian fibration $\underline{\mathcal{P}}(J) \longrightarrow \mathbf{0}_{G}^{op}$, $\underline{\mathcal{P}}(J)_{G/H} = \mathcal{P}(J)^{H}$.

Have two *G*-fixed points, \emptyset and $J \rightarrow$ two cocartesian sections of $\mathcal{P}(J)$, and moreover decompositions $\mathcal{P}(J) \cong \mathbf{O}_{G}^{op} \star_{\mathbf{O}_{G}^{op}} \mathcal{P}(J) \setminus \{\emptyset\} \cong \mathcal{P}(J) \setminus \{J\} \star_{\mathbf{O}_{G}^{op}} \mathbf{O}_{G}^{op}$ Thus, it is sensible to say that a *G*-functor $C \rightarrow D$ is *J*-excisive: it sends *G*-colimit diagrams $\mathcal{P}(J) \rightarrow C$ to *G*-limit diagrams $\mathcal{P}(J) \rightarrow D$. Let J be a finite G-set. G acts on $\mathcal{P}(J)$ by $U \mapsto g \cdot U$. Get cocartesian fibration $\underline{\mathcal{P}}(J) \longrightarrow \mathbf{0}_{G}^{op}$, $\underline{\mathcal{P}}(J)_{G/H} = \mathcal{P}(J)^{H}$.

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Theorem (Dotto-Moi)

 $\mathbf{Sp}^G \simeq \operatorname{Exc}^G_*(\underline{\mathbf{Top}}_{G,*}^{fin}, \underline{\mathbf{Top}}_G)$ where the right-hand side is the full subcategory of $\operatorname{Fun}_G(\underline{\mathbf{Top}}_{G,*}^{fin}, \underline{\mathbf{Top}}_G)$ on the pointed G_+ -excisive functors.

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Naive and wrong idea: Embed C into a S-category $\mathbf{P}_{S}(C)$ s.t. on fibers, $C_{s} \mapsto \operatorname{Fun}(C_{s}^{op}, \operatorname{Top})$. One problem is that we don't have functoriality in the target.

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Vertical opposite: $C^{vop} \longrightarrow S$ is the cocartesian fibration $\Leftrightarrow s \mapsto C_s^{op}$.

Have S-mapping space functor $C^{vop} \times_S C \longrightarrow \mathbf{Top}_{S'}(x, y) \mapsto \operatorname{Map}_{C_s}(x, y)$ (viewed as a functor to **Top**). Adjoint yields S-Yoneda embedding

$$j: C \longrightarrow \mathbf{P}_{\mathcal{S}}(C) := \underline{\operatorname{Fun}}_{\mathcal{S}}(C^{\operatorname{vop}}, \underline{\mathbf{Top}}_{\mathcal{S}}).$$

Have factorization $j_s : C_s \longrightarrow \operatorname{Fun}(C_s^{op}, \operatorname{Top}) \longrightarrow \operatorname{Fun}(C^{vop} \times_S S^{s/}, \operatorname{Top}).$

Let $\underline{\operatorname{Fun}}_{S}^{L}(C, D)$ be the full S-subcategory whose fiber over s consists of those functors that strongly preserve $S^{s/}$ -colimits (= preserve $S^{t/}$ -colimits after pullback by every $S^{t/} \longrightarrow S^{s/}$).

Theorem (S., Riehl–Verity)

 $P_S(C)$ is S-cocomplete, and for any S-cocomplete E, restriction along the S-Yoneda embedding yields an equivalence

$$\underline{\operatorname{Fun}}^{L}_{S}(\mathbf{P}_{S}(C), E) \longrightarrow \underline{\operatorname{Fun}}_{S}(C, E).$$

Proof.

Inverse given by left S-Kan extension along j. Need to identify LHS with those functors which are left S-Kan extensions of their restriction to C.

Letting $C = S = \mathbf{O}_{G}^{op}$, prove Hill's conjecture.

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Thanks for listening!

Image: A matrix and a matrix